

MR5045615 05C20 05C10 05C35

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List orientation number of graphs. (English. English summary)

Discrete Appl. Math. **387** (2026), 225–233.

All graphs considered are finite and simple, and $\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the set of natural numbers. For a graph G , the *orientation number* $o(G)$ is the least natural number k for which G admits an orientation with every out-degree at most k . S. L. Hakimi [J. Franklin Inst. **279** (1965), 290–308; MR0180501] characterized $o(G)$ as the least k with $\text{mad}(G) \leq 2k$, where $\text{mad}(G)$ is the maximum average degree of G ; in particular, $o(G) \leq \lceil \Delta(G)/2 \rceil$ for any graph G . In this paper, the authors introduce a list analogue of the orientation number. Say that a list assignment $L: V(G) \rightarrow \mathcal{P}(\mathbb{N})$ is *valid* if $L(v) \subseteq \{0, 1, \dots, d_G(v)\}$ for every vertex v , and say that G is *L -orientable* if it has an orientation D in which $d_D^+(v) \in L(v)$ for each vertex v . For a natural number k , a graph G is *k -list-orientable* if it is L -orientable for every valid list assignment L such that $|L(v)| \geq \min\{d_G(v) + 1, k\}$ for every vertex v . The *list orientation number*, $o_\ell(G)$, of the graph G is the least k for which G is k -list-orientable. List orientations are complementary to the *F -avoiding orientations* studied by S. Akbari et al. [J. Graph Theory **93** (2020), no. 4, 483–502; MR4061950]—they conjecture (in the language of this paper) that a graph G is L -orientable for any valid list assignment L such that $|L(v)| \geq (d_G(v) + 3)/2$ for every $v \in V(G)$. This conjecture implies that $o_\ell(G) \leq (\Delta(G) + 3)/2$ for every graph G .

In this paper, the authors bound $o_\ell(G)$ in terms of $o(G)$. It is easy to show that $o(G) + 1 \leq o_\ell(G)$ for every graph G , and the authors conjecture that $o_\ell(G) \leq o(G) + 2$ for every graph G . This conjecture implies that $o_\ell(G) \leq \lceil \Delta(G)/2 \rceil + 2$ for every graph G . The main results in this paper are as follows:

- (1) $o_\ell(G) = o(G) + 1$ for any bipartite graph G ;
- (2) if $o(G) \leq 2$, then $o_\ell(G) \leq o(G) + 2$;
- (3) if G is planar, then $o_\ell(G) \leq o(G) + 2$.

The authors also construct examples to show that the upper bounds in (2) and (3) are tight. The main tool is the combinatorial Nullstellensatz; in particular, the authors show that $o_\ell(G) \leq AT(G)$ for any graph G , where $AT(G)$ is the Alon-Tarsi number of G . Lastly, the authors extend their results to orientations of signed graphs (G, σ) , in which a positive edge receives an ordinary orientation while a negative edge is directed towards both of its endpoints or away from both. By way of the signed Alon-Tarsi number $AT(G, \sigma)$ of W. Wang, J. Qian and T. Abe [Graphs Combin. **35** (2019), no. 5, 1051–1064; MR4003656], the bound $o_\ell(G, \sigma) \leq AT(G, \sigma)$ is proved through the combinatorial Nullstellensatz. By a result of Wang, Qian and Abe [op. cit.], $AT(G, \sigma) \leq 5$ for any planar signed graph (G, σ) , so the present authors derive as a corollary that $o_\ell(G, \sigma) \leq 5$ for any planar signed graph (G, σ) .

The O. Henderschedt and J. McDonald preprint cited by the authors has since appeared in print [Discrete Appl. Math. **387** (2026), 116–123; MR5039504]. A few typos are listed below. In the proofs of Claim 2 (page 229) and Claim 4 (page 230), the graph G' is only a subgraph of G , and not a spanning subgraph. In the proof of Claim 2 (page 230), the vertices a_1, b_1, c_1, u_1, v_1 should instead read a_1, d_1, c_1, u_1, v_1 . In the proof of Theorem 4.4 (page 230), the list assignment should be $L(a_i) = L(b_i) = L(c_i) = \{0, 1, 2, 3\}$ for $1 \leq i \leq 7$, not merely for $1 \leq i \leq 3$. In the proof of Claim 5 (page 231), the sum $\sum_i c_D(e_i, v) t_i$ should read $\sum_i c_{D_\sigma}(e_i, v) t_i$. In the same proof (page 232), the final display should read $C||\sigma EE(D)| - |\sigma OE(D)|| \neq 0$. *Brahadeesh Sankarnarayanan*

[References]

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.